IEOR 160: Nonlinear and Discrete Optimization
Univariate Optimization

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This is a summarization of Professor Javad Lavaei's IEOR 160 lecture notes.

1 Global and Local Minimum

Consider a scalar function $f : \mathbb{R} \to \mathbb{R}$. Consider an interval $\mathcal{I} \subset \mathbb{R}$. We want to analyze the problem $\min_{x \in \mathcal{I}} f(x)$. We focus on the minimization case, because if we wish to maximize f, we can simply minimize $-f$.

1. $x_\star$ is a global minimum if $f(x_\star) \leq f(x)$ for all $x \in \mathcal{I}$.
2. $x_\star$ is a strict global minimum if $f(x_\star) < f(x)$ for all $x \in \mathcal{I} \backslash \{x_\star\}$.
3. $x_\star$ is a local minimum if there exists $\delta > 0$ such that $f(x_\star) \leq f(x)$ for all $x \in \mathcal{I} \cap [x_\star - \delta, x_\star + \delta]$.
4. $x_\star$ is a strict local minimum if $\exists \delta > 0$ such that $f(x_\star) < f(x)$ for all $x \in \mathcal{I} \cap [x_\star - \delta, x_\star + \delta] \backslash \{x_\star\}$.
5. A (strict) global minimum is a (strict) local minimum.
6. $x_\star$ may be both a minimum and a maximum (this is true for global case and local case).

2 Univariate Unconstrained Optimization

1. First-order necessary condition (FOC necessary): If $x_\star$ is a local minimum, then $f'(x_\star) = 0$.
2. If $f'(x_\star) = 0$, then $x_\star$ is a stationary point, which may be a local minimum, a local maximum, or a saddle. The FOC itself is insufficient to find the local minimum. We must check the second-order conditions (SOC) to determine the type.
3. SOC necessary: If $x_\star$ is a local minimum, then $f''(x_\star) \geq 0$.
4. SOC sufficient: If $f'(x_\star) = 0$ and $f''(x_\star) > 0$, then $x_\star$ is a strict local minimum.
5. If $f'(x_\star) = 0$ and $f''(x) \geq 0$ for all $x \in \mathbb{R}$, then $x_\star$ is a global minimum.
6. If $f'(x_\star) = 0$ and $f''(x) > 0$ for all $x \in \mathbb{R}$, then $x_\star$ is a strict global minimum.
7. If $f'(x_\star) = f''(x_\star) = 0$, then find the smallest integer k such that $f^{(k)} \neq 0$.

- If k is odd, then $x_\star$ is a saddle.
- If k is even and $f^{(k)} > 0$, then $x_\star$ is a local minimum.
- If k is even and $f^{(k)} < 0$, then $x_\star$ is a local maximum.

3 Univariate Constrained Optimization

Consider the optimization problem $\min_{x\in\mathbb{R}} f(x)$ subject to $a \leq x \leq b$. Any local minimum is either a stationary point or an endpoint of the interval (i.e. $f'(x_\star) = 0$ or $x_\star = a$ or $x_\star = b$).

- At $x_\star = a$, if $f'(a) > 0$, then $x_\star = a$ is a strict local minimum. If $f'(a) < 0$, then $x_\star = a$ is a strict local maximum. If $f'(a) = 0$, then $x_\star = a$ is a stationary point, and we need to check the SOC to determine its type.

- At $x_\star = b$, if $f'(b) < 0$, then $x_\star = b$ is a strict local minimum. If $f'(b) > 0$, then $x_\star = b$ is a strict local maximum. If $f'(b) = 0$, then $x_\star = b$ is a stationary point, and we need to check the SOC to determine its type.

- Between a and b, the optimality condition follows those of the unconstrained case. Make sure to only consider the stationary points between a and b.

When solving this type of problem, we need to check all three cases to find all optimal points.