IEOR 160: Nonlinear and Discrete Optimization
Convex Sets and Convex Functions

Yatong Bai

February 6, 2024

This is a summarization of Professor Javad Lavaei's IEOR 160 lecture notes.

1 Convex Sets

- Affine combination: $\left\{\sum_{i=1}^{k} \alpha_i x_i : \ \sum_{i=1}^{k} \alpha_i = 1\right\}$.
- Convex combination: $\left\{\sum_{i=1}^{k} \alpha_i x_i : \ \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0, \forall i\right\}$.
- A set is affine if for every k points in the set, their affine combination is in the set.
  - A hyperplane is an affine set, but a half-space is not.
- A set is convex if for every k points in the set, their convex combination is in the set.
  - A polyhedron $\left\{x : \ a_i^\top x \leq b_i, c_j^\top x = d, \ \forall i, j\right\}$ is a convex set.
  - Norm balls and half-spaces are convex.
  - The set of PD matrices is convex, and the set of PSD matrices is also convex.
- Convex hull of a set is the smallest convex set containing the set. This can be found by obtaining the convex combination of any k points in the set.
- Operations that preserve convexity:
  - The intersection of convex sets is convex (note that the union of convex sets may not be convex).
  - Affine transformation: consider a convex set $\mathcal{S}$ and an affine function $f : \mathbb{R}^n \to \mathbb{R}^m$. The set $\bar{\mathcal{S}} := \{f(x) : \ x \in \mathcal{S}\}$ is convex.
  - The projections of a convex set are convex.

2 Convex Functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its domain is a convex set and $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$.
  - This is the zeroth-order condition for convexity.

- Replacing $\leq$ with $<$ gives the definition of strict convexity.
- Thus, the set defined by $\{x : f(x) \leq 0\}$, where f is a convex function, is a convex set.

- Convexity does not imply continuity.
  - Example: consider an end point $\tilde{x}$ of dom$f$. f can still be convex if it "jumps up" at $\tilde{x}$.
  - The discontinuity should happen only on the boundaries.

- First-order convexity condition: $f(y) + \nabla f(y)^\top (x - y) \leq f(x)$ for all $x, y \in \text{dom}f$ (replace with $<$ for strict convexity).

- Second-order convexity condition: f is convex if and only if $\nabla^2 f(x) \succeq 0$ for all $x \in \text{dom}f$.
  - If $\nabla^2 f(x) \succ 0$ for all $x \in \text{dom}f$, then f is strictly convex. The reverse direction may not hold (e.g., $f(x) = x^4$).

- Some example convex functions:
  - $f(x) = e^{ax}$.
  - $f(x) = x^a$ where $a \geq 1$ or $a \leq 0$ on $\mathbb{R}_{++}$.
  - $f(x) = -\log(x)$ on $\mathbb{R}_{++}$.
  - Any ℓ_p form function $f(x) = \|x\|_p$.

- Some properties of convex functions:
  - The point-wise maximum of a set of convex functions is convex.
  - If $f(x)$ is convex, then $g(x) = f(Ax + b)$ is also convex.
  - $f(x) := \sum_{i=1}^{k} \alpha_i f_i(x)$ for $a_i \geq 0$ is convex if f_i is convex for all i.

3 Convex Optimization Problems

- Consider an optimization problem $\min_x f(x)$ subject to $x \in \mathcal{X}$. This problem is convex when
  - f is a convex function;
  - $\mathcal{X}$ is a convex set.

- Consider an optimization problem $\min_x f(x)$ subject to $g_i(x) \leq 0$ for all i and $h_j(x) = 0$ for all j. This problem is convex when
  - f is a convex function;
  - g_i is a convex function for each i;
  - h_i is an *affine* function for each j.

- For a convex optimization problem, all local solutions are global.